Options on Hedge Funds under the High-Water Mark Rule

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Abstract

The rapidly growing hedge fund industry has provided individual and institutional investors with new investment vehicles and styles of management. It has also brought forward a new form of performance contract: hedge fund managers receive incentive fees which are typically a fraction of the fund net asset value (NAV) above its starting level - a rule known as *high water mark*.

Options on hedge funds are becoming increasingly popular, in particular because they allow investors with limited capital to get exposure to this new asset class. The goal of the paper is to propose a valuation of plain-vanilla options on hedge funds which accounts for the high water market rule. Mathematically, this valuation leads to an interesting use of local times of Brownian motion. Option prices are numerically computed by inversion of their Laplace transforms.

Keywords: Options on hedge funds; High-water mark; Local time; Excursion theory

I Introduction

The term hedge fund is used to characterize a broad class of "skill-based" asset management firms that do not qualify as mutual funds regulated by the Investment Company Act of 1940 in the United States. Hedge funds are pooled investment vehicles that are privately organized, administered by professional investment managers and not widely available to the general public. Due to their private nature, they carry much fewer restrictions on the use of leverage, short-selling and derivatives than more regulated vehicles.

Across the nineties, hedge funds have been embraced by investors worldwide and are today recognized as an asset class in its own right. Originally, they were operated by taking a "hedged" position against a particular event, effectively reducing the overall risk. Today, the hedge component has totally disappeared and the term "hedge fund" refers to any pooled investment vehicle that is not a conventional fund using essentially long strategies in equity, bonds and money market instruments.

Over the recent years, multi-strategy funds of funds have in turn flourished, providing institutional investors with a whole spectrum of alternative investments exhibiting low correlations with traditional asset classes. In a parallel manner, options on hedge funds have been growing in numbers and types, offering individual investors the possibility of acquiring exposure to hedge funds through a relatively low amount of capital paid upfront at inception of the strategy.

Hedge funds constitute in fact a very heterogeneous group with strategies as diverse as convertible arbitrage, global macro or long short equity. In all cases however, common characteristics may be identified such as long-term commitment of investors, active management and broad discretion granted to the fund manager over the investment style and asset classes selected. Accordingly, incentive fees represent a significant percentage of the performance - typically ranging from 5% to 20%. This performance is most generally measured according to the high-water mark rule, i.e., using as a reference benchmark the Net Asset Value (NAV) of the fund at the time of purchase of the shares or options written on the hedge fund.

So far, the academic literature on hedge funds has focused on such issues as non-normality of returns, actual realized hedge fund performance and persistence of that performance. Amin and Kat (2003) show that, as a stand-alone investment, hedge funds do not offer a superior risk-return profile. Geman and Kharoubi (2003) propose instead the introduction of copulas to better represent the dependence structure between hedge funds and other asset classes. Agarwal and Naik (2000) examine whether persistence is sensitive to the length of the return measurement period and find maximum persistence at a quarterly horizon.

Another stream of papers has analyzed performance incentives in the hedge fund industry (see Fung and Hsieh (1999), Brown, Goetzmann and Ibbotson (1999)). However, the high water mark rule specification has been essentially studied by Goetzman, Ingersoll and Ross (2003).

High-water mark provisions condition the payment of the performance fee upon the hedge fund Net Asset Value exceeding the entry point of the investor. Goetzmann et al examine the costs and benefits to investors of this form of managers' compensation and the consequences of thess option-like characteristics on the values of fees on one hand, investors' claims on the other hand. Our objective is to pursue this analysis one step further and examine the valuation of options on hedge funds under the high-water mark rule. We show that this particular compound option-like problem may be solved in the Black-Scholes (1973) and Merton (1973) setting of geometric Brownian motion for the hedge fund NAV by the use of Local times of Brownian motion.

The remainder of the paper is organized as follows: Section II contains the description of the Net Asset Value dynamics, management and incentive fees and the NAV option valuation. Section II also extends the problem to a moving high water mark. Section III describes numerical examples obtained by inverse Laplace transforms and Monte Carlo simulations. Section IV contains concluding comments.

II The High-Water Mark Rule and Local Times

A. Modeling the High-Water Mark

We work in a continuous-time framework and assume that the fund Net Asset Value (NAV) follows a lognormal diffusion process. This diffusion process will have a different starting point for each investor, depending on the time she entered her position. This starting point will define the high water mark used as the benchmark triggering the performance fees discussed throughout the paper.

We follow Goetzmann, Ingersoll and Ross (2003) in representing the performance fees in the following form

$$f\left(S_t\right) = \mu a \ \mathbf{1}_{\{S_t > H\}} \tag{1}$$

where S_t denotes the Net Asset Value at date t, μ is a mean NAV return statistically observed, a is a percentage generally comprised between 5% and 20% and $H = S_0$ denotes the market value of the NAV as observed at inception of the option contract.

We consider $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P}_0)$ a filtered probability space where $(B_t)_{t\geq 0}$ is an $\{\mathcal{F}_t, t \geq 0\}$ Brownian motion.

We now consider an equivalent measure $\mathbb{Q}$ under which the Net Asset Value dynamics $(S_t)_{t\geq 0}$ satisfy the stochastic differential equation:

$$\frac{dS_t}{S_t} = (r + \alpha - c - f(S_t))dt + \sigma dW_t \tag{2}$$

and the instantaneously compounding interest rate r is supposed to be constant. α denotes the excess return on the fund's assets and is classically defined by

$$\alpha = \mu - r - \beta\,(r_m - r)$$

where r_m is the expected return on the market portfolio. Hence, the "risk-neutral" return on the fund NAV is equal to $(r+\alpha)$[1]; σ denotes the NAV volatility.

The management fees paid regardless of the performance are represented by a constant fraction c (comprised in practice between 0.5% and 2%) of the Net Asset Value. We represent the incentive fees as a deterministic function f of the current value S_t of the NAV, generally chosen according to the high water mark rule defined in equation (1). We can note that management fees have the form of the constant dividend payment of the Merton (1973) model while performance fees may be interpreted as a more involved form of dividend paid to the manager.

Because of their central role in what follows, we introduce the maximum and the minimum processes of the Brownian motion B, namely

$$M_t = \sup_{s\leq t} B_s, \;\; I_t = \inf_{s\leq t} B_s$$

as well as its local time at the level a, $a \in \mathbb{R}$

$$L_t^a = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{|B_s - a| \leq \epsilon\}} ds$$

We also consider $A_t^{(a,+)} = \int_0^t \mathbf{1}_{\{B_s \geq a\}} ds$ and $A_t^{(a,-)} = \int_0^t \mathbf{1}_{\{B_s \leq a\}} ds$, respectively denoting the time spent in $[a;\infty[$ and the time spent in $]-\infty;a]$ by the Brownian motion up to time t.

For simplicity, we shall write $L_t = L_t^0$, $A_t^+ = A_t^{(0,+)}$ and $A_t^- = A_t^{(0,-)}$ the corresponding quantities for $a = 0$.

[1] Our claim is that the measure $\mathbb{Q}$ incorporates the price of market risk as a whole but not the excess performance - the fund "alpha" - achieved by the manager through the selection of specific securities at a given point in time. This view is in agreement with the footnote 6 in Goetzmann, Ingersoll and Ross (2003)

In order to extend our results to different types of incentive fees, we do not specify the function f but only assume that it is a continuous, bounded, increasing and positive function satisfying the following conditions:

$$f(0) = 0, \qquad \lim_{x\to\infty} f(x) < +\infty$$

Proposition II.1 *There exists a unique solution to the stochastic differential equation*

$$\frac{dS_t}{S_t} = (r + \alpha - c - f(S_t))dt + \sigma dW_t$$

Proof Let us denote $Y_t = \frac{\ln(S_t)}{\sigma}$. Applying Itô's formula, we see that the process Y_t satisfies the equation

$$dY_t = dW_t + \psi(e^{\sigma Y_t})dt$$

where $\psi(x) = r - \frac{\sigma^2}{2} + \alpha - c - f(x)$.
f, hence ψ is a Borel bounded function; consequently, we may apply Zvonkin (1974) theorem and obtain strong existence and pathwise uniqueness of the solution of equation (2).
We recall that Zvonkin theorem establishes that for every bounded Borel function ξ, the stochastic differential equation

$$dZ_t = dW_t + \xi(Z_t)dt$$

has a unique solution which is strong, i.e.: in this case, the filtration of Z and W are equal. ■

Integrating equation (2), we observe that this unique solution can be written as

$$S_t = S_0 \exp\left(\left(r + \alpha - c - \frac{\sigma^2}{2}\right)t - \int_0^t f(S_u)du + \sigma W_t\right)$$

We now seek to construct a new probability measure $\mathbb{P}$ under which the expression of S_t reduces to

$$S_t = S_0 \exp(\sigma\widetilde{W}_t) \tag{3}$$

where $\widetilde{W}_t$ is a $\mathbb{P}$ standard Brownian motion.

Proposition II.2 *There exists an equivalent martingale measure $\mathbb{P}$ under which the Net Asset Value dynamics satisfy the stochastic differential equation*

$$\frac{dS_t}{S_t} = \frac{\sigma^2}{2}dt + \sigma d\widetilde{W}_t \tag{4}$$

where

$$\mathbb{Q}_{|\mathcal{F}_t} = Z_t \cdot \mathbb{P}_{|\mathcal{F}_t} \tag{5}$$

$$Z_t = \exp\Big(\int_0^t (b - \frac{f(S_u)}{\sigma}) d\widetilde{W}_u - \frac{1}{2} \int_0^t (b - \frac{f(S_u)}{\sigma})^2 du \Big)$$

and

$$b = \frac{r + \alpha - c - \frac{\sigma^2}{2}}{\sigma}$$

Proof Thanks to Girsanov theorem (see for instance McKean (1969) and Revuz and Yor (2005)) we find that under the probability measure $\mathbb{P}$, $\widetilde{W}_t = W_t + \int_0^t du\big(b - \frac{f(e^{\sigma Y_u})}{\sigma}\big)$ is a Brownian motion, which allows us to conclude. ■

B. Building the Pricing Framework

For practical purposes, the issuer of the call is typically the hedge fund itself, hence hedging arguments allow to price the option as the expectation (under the right probability measure) of the discounted payoff. More generally, a European-style hedge fund derivative with maturity $T > 0$ is defined by its payoff $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and the valuation of the option reduces to computing expectations of the following form:

$$V_F(t, S, T) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \big[F(S_u; u \leq T) \big| \mathcal{F}_t \big]$$

For the case where the valuation of the option takes place at a date $t = 0$, we denote $V_F(S, T) = V_F(0, S, T)$. We can observe that we are in a situation of complete markets since the only source of randomness is the Brownian motion driving the NAV dynamics.

Proposition II.3 *For any payoff F that can be written as an increasing function of the stock price process, the option price associated to the above payoff is an increasing function of the high-water mark level.*

Proof This result is quite satisfactory from a financial perspective. Mathematically, it may be deduced from the following result :
Let us consider the solutions (S^1, S^2) of the pair of stochastic differential equations :

$$\begin{aligned} dS_t^1 &= b^1(S_t^1)dt + \sigma S_t^1 dW_t \\ dS_t^2 &= b^2(S_t^2)dt + \sigma S_t^2 dW_t \end{aligned}$$

where

$$\begin{aligned}b^1(x) &= (r+\alpha-c-\mu a\ \mathbf{1}_{\{x>H\}})x\\ b^2(x) &= (r+\alpha-c-\mu a\ \mathbf{1}_{\{x>H'\}})x\end{aligned}$$

with $H > H'$ and $S_0^1 = S_0^2$ a.s.
We may apply a comparison theorem since b^1 and b^2 are bounded Borel functions and $b^1 \geq b^2$ everywhere, obtain that

$$\mathbb{P}[S_t^1 \geq S_t^2; \forall t \geq 0] = 1$$

and then conclude. ■

If we consider a call option and a put option with strike K and maturity T, we observe the following call-put parity relation:

$$C_0(K,T) - P_0(K,T) = \mathbb{E}^{\mathbb{Q}}[e^{-rT}S_T] - Ke^{-rT} \tag{6}$$

We now wish to express the exponential $(\mathcal{F}_t, \mathbb{P})$-martingale Z_t featured in (5) in terms of well-known processes in order to be able to obtain closed-form pricing formulas.

Lemma II.4 *Let us define d_H, λ, α_+, α_- and ϕ as follows:*

$$d_H = \frac{\ln(\frac{H}{S_0})}{\sigma}, \qquad \lambda = \frac{\mu a}{2\sigma}$$

$$\alpha_+ = 2\lambda^2 + \frac{b^2}{2} - 2\lambda b, \qquad \alpha_- = \frac{b^2}{2}$$

$$\phi(x) = e^{bx - 2\lambda(x-d_H)_+}$$

We then obtain:

$$Z_t = e^{2\lambda(-d_H)_+}\phi(\widetilde{W}_t)\exp(\lambda L_t^{d_H})\exp(-\alpha_+ A_t^{(d_H,+)} - \alpha_- A_t^{(d_H,-)}) \tag{7}$$

Proof The proof of this proposition is based on the one hand on the Tanaka formula which, for a Brownian motion B and any real number a, establishes that

$$(B_t - a)_+ = (-a)_+ + \int_0^t dB_s \mathbf{1}_{\{B_s > a\}} + \frac{1}{2}L_t^a$$

On the other hand, we can rewrite

$$f(S_t) = \mu a \mathbf{1}_{\{\widetilde{W}_t > d_H\}}$$

Observing that $A_t^{(d_H,+)} + A_t^{(d_H,-)} = t$ leads to the result. ■

From the above lemma, we obtain that:

$$\begin{aligned} V_F(S,T) &= e^{-rT}\mathbb{E}^{\mathbb{P}}\big[Z_T F(S_u; u \le T)\big] \\ &= e^{-rT+2\lambda(-d_H)_+}\mathbb{E}^{\mathbb{P}}\big[\phi(W_T)\exp(\lambda L_T^{d_H} - \alpha_+ A_T^{(d_H,+)} - \alpha_- A_T^{(d_H,-)})F(S_0 e^{\sigma \widetilde{W}_u}; u \le T)\big] \end{aligned}$$

The price of a NAV call option is closely related to the law of the triple $(W_t, L_t^a, A_t^{(a,+)})$. Karatzas and Shreve (1991) have extensively studied this joint density for $a = 0$ and obtained in particular the following remarkable result

Proposition II.5 *For any positive t and b, $0 < \tau < t$, we have*

$$\begin{aligned} \mathbb{P}[W_t \in dx; L_t \in db, A_t^+ \in d\tau] &= f(x, b; t, \tau)\ dx\ db\ d\tau; \qquad x > 0 \\ &= f(-x, b; t, -\tau)\ dx\ db\ d\tau; \qquad x < 0 \end{aligned}$$

where

$$f(x, b; t, \tau) = \frac{b(2x+b)}{8\pi\tau^{\frac{3}{2}}(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{b^2}{8(t-\tau)} - \frac{(2x+b)^2}{8\tau} \right)$$

This formula could lead to a computation of the option price based on a multiple integration but it would be numerically intensive; moreover, obtaining an analytical formula for the triple integral involved in the option price seems quite unlikely. We observe instead that in the above density f, a convolution product appears, which leads us to compute either Fourier or Laplace transforms. We are in fact going to compute the Laplace transform with respect to time to maturity of the option price. This way to proceed is mathematically related to the Karatzas and Shreve result in Proposition II.5. In the same way, we can notice that the Laplace transform exhibited by Geman and Yor (1996) for the valuation of a Double Barrier option is related to the distribution of the triple (W_t, M_t, I_t) Brownian motion, its maximum and minimum used by Kunitomo and Ikeda (1992) for the same pricing problem. The formulas involved in the NAV call price rely on the following result which may be obtained from Brownian excursion theory:

Proposition II.6 *Let W_t be a standard Brownian motion, L_t its local time at zero, A_t^+ and A_t^- the times spent positively and negatively until time t.*

For any function $h \in L^1(\mathbb{R})$, the Laplace transform of the quantity $g(t) = \mathbb{E}\big[h(W_t)\exp(\lambda L_t)\exp(-\mu A_t^+ - \nu A_t^-)\big]$ has the following analytical expression

$$\int_0^\infty dt e^{-\frac{\theta}{2}t} g(t) = 2\frac{\left(\int_0^\infty dx e^{-x\sqrt{\theta+2\mu}} h(x) + \int_0^\infty dx e^{-x\sqrt{\theta+2\nu}} h(-x)\right)}{\sqrt{\theta+2\mu} + \sqrt{\theta+2\nu} - 2\lambda}$$

for θ large enough to ensure positivity of the denominator.

Proof See the Appendix for details. The result is rooted in the theory of excursions of the Brownian motion. ■

C. Valuation of the Option at Inception of the Contract

In this section, we turn to the computation of the price of a European call option written on a Hedge Fund NAV under the high-water mark rule. Consequently, the payoff considered is the following:

$$F(S_u; u \leq T) = (S_T - K)_+ \tag{8}$$

or, in a more convenient way for our purpose

$$F(\widetilde{W}_u; u \leq T) = (S_0 \exp(\sigma \widetilde{W}_T) - K)_+$$

At inception of the contract, the high-water mark that is chosen is the spot price, hence $H = S_0$ and $d_H = 0$. This specific framework allows us to use fundamental results on the joint law of the triple (B_t, L_t^0, A_t^+) presented in Proposition II.6. We write the European call option price as follows

$$C(0, S_0) = e^{-rT}\mathbb{E}^{\mathbb{P}}\big[h(\widetilde{W}_T)\exp(\lambda L_T - \alpha_+ A_T^+ - \alpha_- A_T^-)\big]$$

where $h(x) = (S_0 e^{\sigma x} - K)_+ e^{bx - 2\lambda(x)_+}$.

We now compute the Laplace transform in time to maturity of the European call option on the NAV of an Hedge Fund, that is to say the following quantity:

$$\begin{aligned} \forall \theta \in \mathbb{R}_+ \qquad I(\theta) &= \int_0^\infty dt e^{-\frac{\theta}{2}t} e^{-rt}\mathbb{E}^{\mathbb{Q}}[(S_t - K)_+] \\ &= \int_0^\infty dt e^{-(\frac{\theta}{2}+r)t}\mathbb{E}^{\mathbb{P}}[Z_t(S_t - K)_+] \end{aligned}$$

Lemma II.7 *The Laplace transform with respect to time to maturity of a call option price has the following analytical expression:*

$$I(\theta) = 2\frac{\left(\int_0^\infty dx e^{-x\sqrt{\theta+2(r+\alpha_+)}}h(x) + \int_0^\infty dx e^{-x\sqrt{\theta+2(r+\alpha_-)}}h(-x)\right)}{\sqrt{\theta+2(r+\alpha_+)}+\sqrt{\theta+2(r+\alpha_-)}-2\lambda} \tag{9}$$

where $h(x) = e^{bx-2\lambda x_+}(S_0 e^{\sigma x} - K)_+$

Proof We obtain from Lemma II.4 that:

$$\mathbb{E}^{\mathbb{P}}[Z_t(S_t-K)_+] = \mathbb{E}\big[h(\widetilde{W_t})\exp(\lambda L_t)\exp(-\alpha_+ A_t^+ - \alpha_- A_t^-)\big]$$

where:

$$h(x) = e^{bx-2\lambda x_+}(S_0 e^{\sigma x} - K)_+$$

Then, using Proposition II.6, we are able to conclude. ■

This lemma leads us to compute explicit formulas for the Laplace transform of a call option that is in-the-money ($S_0 \geq K$) at date 0 and out-of-the-money ($S_0 < K$) that we present in two consecutive propositions.

Proposition II.8 *For an out-of-the-money call option ($S_0 \leq K$), the Laplace transform of the price is given by the following formula:*

$$I(\theta) = \frac{N(\theta)}{D(\theta)}$$

where

$$\theta > (\sigma + b - 2\lambda)^2 - 2(r+\alpha_+)$$

and

$$\begin{aligned}
D(\theta) &= \frac{\sqrt{\theta+2(r+\alpha_+)}+\sqrt{\theta+2(r+\alpha_-)}-2\lambda}{2} \\
N(\theta) &= \frac{S_0}{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-\sigma-b}\left(\frac{S_0}{K}\right)^{\frac{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-\sigma-b}{\sigma}} \\
&\quad - \frac{K}{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-b}\left(\frac{S_0}{K}\right)^{\frac{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-b}{\sigma}}
\end{aligned}$$

Proof Keeping the notation of Proposition II.6, we can write

$$\forall x > 0, \qquad h(x) = (S_0 e^{\sigma x} - K)\mathbf{1}_{\{x \geq \frac{1}{\sigma}\ln(\frac{K}{S_0})\}} e^{(b-2\lambda)x} \qquad and \qquad h(-x) = 0$$

and then by simple integration, obtain the stated formula. ■

Proposition II.9 *For an in-the-money call option ($S_0 \geq K$), the Laplace transform of the price is given by the following formula:*

$$I(\theta) = \frac{N_1(\theta) + N_2(\theta)}{D(\theta)}$$

where $\theta > (\sigma + b - 2\lambda)^2 - 2(r + \alpha_+)$ and

$$\begin{aligned}
D(\theta) &= \frac{\sqrt{\theta + 2(r+\alpha_+)} + \sqrt{\theta + 2(r+\alpha_-)} - 2\lambda}{2} \\
N_1(\theta) &= \frac{S_0}{\sqrt{\theta + 2(r+\alpha_+)} + 2\lambda - \sigma - b} - \frac{K}{\sqrt{\theta + 2(r+\alpha_+)} + 2\lambda - b} \\
N_2(\theta) &= \frac{S_0}{\sqrt{\theta + 2(r+\alpha_-)} + \sigma + b}\left(1 - (\frac{K}{S_0})^{\frac{\sqrt{\theta+2(r+\alpha_-)}+\sigma+b}{\sigma}}\right) \\
&\quad - \frac{K}{\sqrt{\theta + 2(r+\alpha_-)} + b}\left(1 - (\frac{K}{S_0})^{\frac{\sqrt{\theta+2(r+\alpha_-)}+b}{\sigma}}\right),
\end{aligned}$$

α_- and α_+ being defined in Lemma II.4 .

Proof We have

$$\forall x > 0, \qquad h(x) = (S_0 e^{\sigma x} - K)e^{(b-2\lambda)x} \qquad and \qquad h(-x) = (S_0 e^{-\sigma x} - K)\mathbf{1}_{\{x \leq \frac{1}{\sigma}\ln(\frac{S_0}{K})\}} e^{-bx}$$

and as in the previous proposition, the Laplace transform is derived. ■

As a side note, we observe that the case $K = 0$ provides the Laplace transform of the $t-$maturity forward contract written on the NAV at date 0

$$\int_0^\infty dt e^{-\frac{\theta}{2}t}\mathbb{E}^{\mathbb{P}}[e^{-rt}S_t] = 2\frac{\frac{S_0}{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-\sigma-b} + \frac{S_0}{\sqrt{\theta+2(r+\alpha_-)}+\sigma+b}}{\sqrt{\theta + 2(r+\alpha_+)} + \sqrt{\theta + 2(r+\alpha_-)} - 2\lambda}$$

where $\theta > (\sigma + b - 2\lambda)^2 - 2(r + \alpha_+)$.

It is satisfactory to check that by choosing $a = 0$, $\alpha = 0$, we obtain the Laplace transform of a European call option on a dividend-paying stock with a continuous dividend yield c whose dynamics satisfy as in Merton (1973), the equation

$$\frac{dS_t}{S_t} = (r - c)\ dt + \sigma\ dW_t$$

This Laplace transform is derived from Proposition II.8 for an out-of-the-money call option and from Proposition II.9 for an in-the-money call option.

D. Valuation during the lifetime of the Option

Evaluating at a time t a call option on a hedge fund written at date 0 implies that we are in the situation where $d_H = \frac{1}{\sigma}\ln(\frac{H}{S_t})$ may be different from 0. Since the solution of the stochastic differential equation driving the Net Asset Value is a Markov process, the evaluation of the option at time t only depends on the value of the process at time t and on the time to maturity $T - t$. Hence, we need to compute the following quantity

$$C(t, S_t) = \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)_+|\mathcal{F}_t\big]$$

Given the relationship between $\mathbb{P}$ and $\mathbb{Q}$, we can write

$$C(t, S_t) = e^{-r(T-t)}e^{2\lambda(-d_H)_+}\mathbb{E}^{\mathbb{P}}\big[h(\widetilde{W}_{T-t})\exp(\lambda L^{d_H}_{T-t} - \alpha_+ A^{(d_H,+)}_{T-t} - \alpha_- A^{(d_H,-)}_{T-t})\big]$$

where $h(x) = e^{bx-2\lambda(x-d_H)_+}(S_t e^{\sigma x} - K)_+$
Because of the importance of the level d_H in the computations, we introduce the stopping time $\tau_{d_H} = \inf\{t \geq 0; \widetilde{W}_t = d_H\}$ and split the problem into the computation of the two following quantities:

$$C_1 = e^{-r(T-t)}e^{2\lambda(-d_H)_+}\mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{\tau_{d_H} > T-t\}}h(\widetilde{W}_{T-t})\exp(\lambda L^{d_H}_{T-t} - \alpha_+ A^{(d_H,+)}_{T-t} - \alpha_- A^{(d_H,-)}_{T-t})\big]$$

and

$$C_2 = e^{-r(T-t)}e^{2\lambda(-d_H)_+}\mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{\tau_{d_H} < T-t\}}h(\widetilde{W}_{T-t})\exp(\lambda L^{d_H}_{T-t} - \alpha_+ A^{(d_H,+)}_{T-t} - \alpha_- A^{(d_H,-)}_{T-t})\big]$$

In order to compute C_1, we introduce for simplicity $s = T - t$ and obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{\tau_{d_H} > s\}}h(\widetilde{W}_s)e^{\lambda L^{d_H}_s - \alpha_+ A^{(d_H,+)}_s - \alpha_- A^{(d_H,-)}_s}\big] &= e^{-s\alpha_-}\mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{M_s < d_H\}}h(\widetilde{W}_s)\big] \quad \text{if} \quad d_H > 0 \\ &= e^{-s\alpha_+}\mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{I_s > d_H\}}h(\widetilde{W}_s)\big] \quad \text{if} \quad d_H < 0\end{aligned}$$

We now need to recall some well-known results on Brownian motion first-passage times that one may find for instance in Karatzas and Shreve (1991).

Lemma II.10 *The following equalities hold for* $u > 0$ *and* $a > 0$

$$\mathbb{P}[\tau_a \leq u] = \mathbb{P}[M_u \geq a] = \frac{2}{\sqrt{2\pi}} \int_{\frac{a}{\sqrt{u}}}^{\infty} e^{-\frac{x^2}{2}} dx$$

Hence, for $u > 0$ *and* $a \in \mathbb{R}$

$$\mathbb{P}[\tau_a \in du] = \frac{|a|}{\sqrt{2\pi u^3}} e^{-\frac{a^2}{2u}} du$$

and for $\lambda > 0$

$$\mathbb{E}[e^{-\lambda \tau_a}] = e^{-|a|\sqrt{2\lambda}}$$

where $\tau_a = \inf\{t \geq 0; W_t = a\}$

Lemma II.11 *For* $b \geq 0$ *and* $a \leq b$*, the joint density of* (W_u, M_u) *is given by :*

$$\mathbb{P}[W_u \in da, M_u \in db] = \frac{2(2b-a)}{\sqrt{2\pi u^3}} \exp\Big\{-\frac{(2b-a)^2}{2u}\Big\} da\ db$$

and likewise, for $b \leq 0$ *and* $a \geq b$ *the joint density of* (W_u, I_u) *is given by*

$$\mathbb{P}[W_u \in da, I_u \in db] = \frac{2(a-2b)}{\sqrt{2\pi u^3}} \exp\Big\{-\frac{(2b-a)^2}{2u}\Big\} dadb$$

These lemmas provide us with the following interesting property

Proposition II.12 *Let us consider* W_u *a standard Brownian motion,* I_u *and* M_u *respectively its minimum and maximum values up to time* u*.*
For any function $h \in L^1(\mathbb{R})$*, the quantity* $k_a(u) = \mathbb{E}\big[\mathbf{1}_{\{\tau_a > u\}} h(W_u)\big]$ *is given by*

$$\int_{-\infty}^{\frac{a}{\sqrt{u}}} dv \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} h(v\sqrt{u}) - \int_{-\infty}^{-\frac{a}{\sqrt{u}}} dv \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} h(v\sqrt{u} + 2a) \qquad \textit{if} \quad a > 0$$

$$\int_{-\infty}^{-\frac{a}{\sqrt{u}}} dv \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} h(-v\sqrt{u}) - \int_{-\infty}^{\frac{a}{\sqrt{u}}} dv \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} h(-v\sqrt{u} + 2a) \qquad \textit{if} \quad a < 0$$

Proof We first observe that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{\tau_a > u\}} h(W_u)\big] &= \mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{M_u < a\}} h(W_u)\big] \quad \text{if} \quad a > 0 \\ &= \mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{I_u > a\}} h(W_u)\big] \quad \text{if} \quad a < 0 \end{aligned}$$

By symmetry, we only need to show the result in the case $a > 0$. From the previous lemma, we can write

$$\mathbb{E}\big[\mathbf{1}_{\{M_u<a\}}h(W_u)\big] = \int_0^a db \int_{-\infty}^b dx h(x) \frac{2(2b-x)}{\sqrt{2\pi u^3}} \exp\big\{-\frac{(2b-x)^2}{2u}\big\}$$

Finally, we conclude by applying Fubini's theorem. ■

As a consequence, we can now compute the quantity C_1

Proposition II.13 *For a call option such that $d_H > 0$ or equivalently $H > S_t$, the quantity C_1 is equal to*

$$e^{-(r+\alpha_-)s} G(K, H, S_t, s)$$

where $s = T - t$ and

$$\begin{aligned}
G(K,H,S_t,s) &= 0 \quad if \quad K \geq H \\
G(K,H,S_t,s) &= S_t e^{s\frac{(b+\sigma)^2}{2}} N_1 - K e^{s\frac{b^2}{2}} N_2 \quad if \quad K < H \\
N_1 &= N(\frac{d_H}{\sqrt{s}} - \sqrt{s}(b+\sigma)) - N(\frac{d_K}{\sqrt{s}} - \sqrt{s}(b+\sigma)) \\
&\quad - e^{2(b+\sigma)d_H}\big(N(-\frac{d_H}{\sqrt{s}} - \sqrt{s}(b+\sigma)) - N(\frac{d_K - 2d_H}{\sqrt{s}} - \sqrt{s}(b+\sigma))\big) \\
N_2 &= N(\frac{d_H}{\sqrt{s}} - \sqrt{s}b) - N(\frac{d_K}{\sqrt{s}} - \sqrt{s}b) \\
&\quad - e^{2bd_H}\big(N(-\frac{d_H}{\sqrt{s}} - \sqrt{s}b) - N(\frac{d_K - 2d_H}{\sqrt{s}} - \sqrt{s}b)\big)
\end{aligned}$$

where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x dy e^{-\frac{y^2}{2}}$

Proof We apply Proposition II.12 in the case $a > 0$ with $h(x) = (S_0 e^{\sigma x} - K)_+ e^{bx - 2\lambda(x-d_H)_+}$. ■

Proposition II.14 *For a call option such that $d_H < 0$ or equivalently $H < S_t$, the quantity C_1 is given by*

$$e^{-(r+\alpha_+)s} J(K, H, S_t, s)$$

where $s = T - t$

$$
\begin{aligned}
J(K, H, S_t, s) &= S_t e^{s\frac{(b-2\lambda+\sigma)^2}{2}} N_1(d_1, d_2) - K e^{s\frac{(b-2\lambda)^2}{2}} N_2(d_1, d_2) \\
N_1(d_1, d_2) &= N(-\frac{d_1}{\sqrt{s}} + \sqrt{s}(b + \sigma - 2\lambda)) - e^{2(b+\sigma-2\lambda)d_H} N(\frac{d_2}{\sqrt{s}} + \sqrt{s}(b + \sigma - 2\lambda)) \\
N_2(d_1, d_2) &= N(-\frac{d_1}{\sqrt{s}} + \sqrt{s}(b - 2\lambda)) - e^{2(b-2\lambda)d_H} N(\frac{d_2}{\sqrt{s}} + \sqrt{s}(b - 2\lambda)) \\
(d_1, d_2) &= (d_K, 2d_H - d_K) \quad if \quad K > H \\
(d_1, d_2) &= (d_H, d_H) \quad if \quad K \le H
\end{aligned}
$$

where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dy e^{-\frac{y^2}{2}}$

Proof We apply Proposition II.12 in the case $a < 0$ with $h(x) = (S_0 e^{\sigma x} - K)_+ e^{bx - 2\lambda(x - d_H)_+}$. ■

In order to compute C_2, it is useful to exhibit a result similar to the one obtained in Proposition II.5 to obtain the Laplace transform of the joint density of $(B_t, L_t^a, A_t^{(a,+)}, A_t^{(a,-)})$.

Proposition II.15 *Let us consider W_t a standard Brownian motion, L_t^a its local time at the level a, $A_t^{(a,+)}$ and $A_t^{(a,-)}$ respectively the time spent above and below a by the Brownian motion W until time t.*
For any function $h \in L^1(\mathbb{R})$, the Laplace Transform $\int_0^\infty dt e^{-\frac{\theta}{2}t} g_a(t)$ of the quantity $g_a(t) = \mathbb{E}\big[\mathbf{1}_{\{\tau_a < t\}} h(W_t) \exp(\lambda L_t^a) \exp(-\mu A_t^{(a,+)} - \nu A_t^{(a,-)})\big]$ is given by

$$
\begin{aligned}
&2e^{-a\sqrt{\theta+2\nu}} \frac{\left(\int_0^\infty dx e^{-x\sqrt{\theta+2\mu}} h(a+x) + \int_0^\infty dx e^{-x\sqrt{\theta+2\nu}} h(a-x)\right)}{\sqrt{\theta+2\mu} + \sqrt{\theta+2\nu} - 2\lambda} \qquad if \quad a > 0 \\
&2e^{a\sqrt{\theta+2\mu}} \frac{\left(\int_0^\infty dx e^{-x\sqrt{\theta+2\mu}} h(a+x) + \int_0^\infty dx e^{-x\sqrt{\theta+2\nu}} h(a-x)\right)}{\sqrt{\theta+2\mu} + \sqrt{\theta+2\nu} - 2\lambda} \qquad if \quad a < 0
\end{aligned}
$$

for θ large enough, as seen before.

Proof Let us prove this result in the case $a > 0$; it easily yields to the case

$a < 0$.
We first write

$$g_a(t) = e^{-\nu t}\mathbb{E}\big[\mathbf{1}_{\{\tau_a<t\}}h(W_t)\exp(\lambda L_t^a)e^{-(\mu-\nu)A_t^{(a,+)}}\big]$$

Then

$$I(\theta) = \int_0^{+\infty} dt\ e^{-t\frac{\theta+2\nu}{2}}\mathbb{E}\big[\mathbf{1}_{\{\tau_a<t\}}h(W_t)\exp(\lambda L_t^a)e^{-(\mu-\nu)A_t^{(a,+)}}\big]$$

We now use the strong Markov property and observe that $B_t = W_{t+\tau_a} - W_{\tau_a} = W_{t+\tau_a} - a$ is a Brownian motion. Next, we compute the quantity

$$\begin{aligned}
\mathbb{E}\big[\mathbf{1}_{\{\tau_a<t\}}h(W_t)\exp(\lambda L_t^a)e^{-(\mu-\nu)A_t^{(a,+)}}\big] &= \mathbb{E}\big[\mathbf{1}_{\{\tau_a<t\}}h(B_{t-\tau_a}+a)\exp(\lambda L_{t-\tau_a})e^{-(\mu-\nu)A_{t-\tau_a}^{+}}\big] \\
&= \int_0^t ds\frac{ae^{-\frac{a^2}{2s}}}{\sqrt{2\pi s^3}}\mathbb{E}\big[h(B_{t-s}+a)\exp(\lambda L_{t-s})e^{-(\mu-\nu)A_{t-s}^{+}}\big] \\
&= \int_0^t ds\frac{ae^{-\frac{a^2}{2(t-s)}}}{\sqrt{2\pi (t-s)^3}}\mathbb{E}\big[h(B_{s}+a)\exp(\lambda L_{s})e^{-(\mu-\nu)A_{s}^{+}}\big]
\end{aligned}$$

since

$$\begin{aligned}
L_t^a(a + B_{(\cdot-\tau_a)_+}) &= L_{(t-\tau_a)_+} \\
A_t^{(a,+)} = \int_0^t ds\mathbf{1}_{\{B_{(s-\tau_a)_+}>0\}} &= A^+_{(t-\tau_a)_+}
\end{aligned}$$

Hence, applying Fubini's theorem and Proposition II.6 we obtain

$$\begin{aligned}
I(\theta) &= \int_0^\infty dse^{-\frac{\theta}{2}s}\mathbb{E}\big[h(a+B_s)\exp(\lambda L_s)\exp(-\mu A_s^+ - \nu A_s^-)\big]\int_0^\infty due^{-\frac{\theta+2\nu}{2}u}\frac{|a|e^{-\frac{a^2}{2u}}}{\sqrt{2\pi u^3}} \\
&= 2e^{-a\sqrt{\theta+2\nu}}\frac{\left(\int_0^\infty dxe^{-x\sqrt{\theta+2\mu}}h(a+x) + \int_0^\infty dxe^{-x\sqrt{\theta+2\nu}}h(a-x)\right)}{\sqrt{\theta+2\mu}+\sqrt{\theta+2\nu}-2\lambda}
\end{aligned}$$

■

Proposition II.16 *In the case $H \leq K$, the Laplace transform with respect to the variable $T-t$ of the quantity C_2 is given by the following formula:*

$$I(\theta) = M(\theta)\frac{N(\theta)}{D(\theta)}$$

where

$$\theta > (\sigma + b - 2\lambda)^2 - 2(r + \alpha_+)$$

and

$$\begin{aligned}
M(\theta) &= \left(\frac{H}{S_t}\right)^{\frac{b-\sqrt{\theta+2(r+\alpha_-)}}{\sigma}} \qquad if \quad H > S_t \\
M(\theta) &= \left(\frac{S_t}{H}\right)^{\frac{2\lambda-b-\sqrt{\theta+2(r+\alpha_+)}}{\sigma}} \qquad if \quad H < S_t \\
D(\theta) &= \frac{\sqrt{\theta+2(r+\alpha_+)} + \sqrt{\theta+2(r+\alpha_-)} - 2\lambda}{2} \\
N(\theta) &= \frac{H}{\sqrt{\theta+2(r+\alpha_+)} + 2\lambda - \sigma - b}\left(\frac{H}{K}\right)^{\frac{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-\sigma-b}{\sigma}} \\
&\quad - \frac{K}{\sqrt{\theta+2(r+\alpha_+)} + 2\lambda - b}\left(\frac{H}{K}\right)^{\frac{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-b}{\sigma}}
\end{aligned}$$

Proof We prove this result by applying Proposition II.8 and Proposition II.15 and noticing that $(S_0e^{\sigma(x+d_H)} - K)_+ = (He^{\sigma x} - K)_+$ ■

Proposition II.17 *In the case $H \geq K$, the Laplace transform with respect to the variable $T - t$ of the quantity C_2 is given by the formula:*

$$\begin{aligned}
I(\theta) &= M(\theta)\frac{N_1(\theta) + N_2(\theta)}{D(\theta)} \\
M(\theta) &= \left(\frac{H}{S_t}\right)^{\frac{b-\sqrt{\theta+2(r+\alpha_-)}}{\sigma}} \qquad if \quad H > S_t \\
M(\theta) &= \left(\frac{S_t}{H}\right)^{\frac{2\lambda-b-\sqrt{\theta+2(r+\alpha_+)}}{\sigma}} \qquad if \quad H < S_t \\
D(\theta) &= \frac{\sqrt{\theta+2(r+\alpha_+)} + \sqrt{\theta+2(r+\alpha_-)} - 2\lambda}{2} \\
N_1(\theta) &= \frac{H}{\sqrt{\theta+2(r+\alpha_+)} + 2\lambda - \sigma - b} - \frac{K}{\sqrt{\theta+2(r+\alpha_+)} + 2\lambda - b} \\
N_2(\theta) &= \frac{H}{\sqrt{\theta+2(r+\alpha_-)} + \sigma + b}\left(1 - (\frac{K}{H})^{\frac{\sqrt{\theta+2(r+\alpha_-)}+\sigma+b}{\sigma}}\right) \\
&\quad - \frac{K}{\sqrt{\theta+2(r+\alpha_-)} + b}\left(1 - (\frac{K}{H})^{\frac{\sqrt{\theta+2(r+\alpha_-)}+b}{\sigma}}\right)
\end{aligned}$$

where $\theta > (\sigma + b - 2\lambda)^2 - 2(r + \alpha_+)$

Proof This result is immediately derived from Proposition II.9 and Proposition II.15. ■

E. Extension to a Moving High-Water Mark

We now wish to take into account the fact that the threshold triggering the performance fees may accrue at the risk-free rate. As a consequence, we define $\widetilde{f}$ as

$$\widetilde{f}(t, S_t) = \mu a \mathbf{1}_{\{S_t > He^{rt}\}}$$

Proposition II.18 *There exists a unique solution to the stochastic differential equation*

$$\frac{dS_t}{S_t} = (r + \alpha - c - \widetilde{f}(t, S_t))dt + \sigma dW_t \tag{10}$$

Proof Let us denote $Y_t = \frac{\ln(S_t e^{-rt})}{\sigma}$. Applying Itô's formula, we can see that Y_t satisfies the following equation

$$dY_t = dW_t + \psi(e^{\sigma Y_t})dt$$

where $\psi(x) = -\frac{\sigma^2}{2} + \alpha - c - f(x)$ and f denotes the performance fees function defined in equation (1).
ψ is Borel locally bounded, consequently we may again apply Zvonkin theorem that ensures strong existence and pathwise uniqueness of the solution of (10). ■

Let us denote $\widetilde{S}_t = S_t e^{-rt}$; we seek to construct a probability measure $\widehat{\mathbb{Q}}$ under which

$$\widetilde{S}_t = S_0 \exp(\sigma \widehat{W}_t)$$

where $\widehat{W}_t$ is a $\widehat{\mathbb{Q}}$ standard Brownian motion. We briefly extend the results of the previous section to the case of a moving high-water mark.

Proposition II.19 *There exists an equivalent martingale measure* $\widehat{\mathbb{Q}}$ *under which the Net Asset Value dynamics satisfy the stochastic differential equation*

$$\frac{dS_t}{S_t} = (r + \frac{\sigma^2}{2})dt + \sigma d\widehat{W}_t \tag{11}$$

Moreover,

$$\mathbb{Q}_{|\mathcal{F}_t} = Z_t \cdot \widehat{\mathbb{Q}}_{|\mathcal{F}_t} \tag{12}$$

where

$$Z_t = \exp\Big(\int_0^t \big(b - \frac{f(\widetilde{S}_u)}{\sigma}\big) d\widehat{W}_u - \frac{1}{2}\int_0^t \big(b - \frac{f(\widetilde{S}_u)}{\sigma}\big)^2 du\Big)$$

and

$$b = \frac{\alpha - c - \frac{\sigma^2}{2}}{\sigma}$$

Lemma II.20 *Let us define d_H, λ, α_+, α_- and ϕ as follows:*

$$d_H = \frac{\ln(\frac{H}{S_0})}{\sigma}, \qquad \lambda = \frac{\mu a}{2\sigma}$$

$$\alpha_+ = 2\lambda^2 + \frac{b^2}{2} - 2\lambda b, \qquad \alpha_- = \frac{b^2}{2}$$

$$\phi(x) = e^{bx - 2\lambda(x-d_H)_+}$$

We then obtain:

$$Z'_t = e^{2\lambda(-d_H)_+}\phi(\widehat{W}_t)\exp(\lambda L_t^{d_H})\exp(-\alpha_+ A_t^{(d_H,+)} - \alpha_- A_t^{(d_H,-)}) \tag{13}$$

For the sake of simplicity, we write in this paragraph the strike as Ke^{rT} and need to compute

$$C(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\big[(S_T - Ke^{rT})_+ | \mathcal{F}_t\big] \tag{14}$$

The pricing formulas[2] are derived in a remarkably simple manner by setting $r = 0$ in the results obtained in II.C and II.D.

III Numerical Approaches to the NAV option prices

At this point, we are able to compute option prices thanks to Laplace Transforms techniques (see Abate and Whitt (1995)) or Fast Fourier Transforms techniques (see Walker (1996)) . We can observe that if Monte Carlo simulations were performed in order to obtain the NAV option price, the number of such simulations would be fairly large because of the presence of an indicator variable in the Net Asset Value dynamics. The computing time involved

[2]All full proofs may be obtained from the authors.

in the inversion of Laplace transforms is remarkably lower compared to the one attached to Monte Carlo simulations. The times to maturity considered below are chosen to be less or equal to one year in order to avoid the high water mark reset arising for more distant maturities. Taking into account the reset feature would lead to computations analogous to the ones involved in forward start options and is not the primary focus of this paper.
Tables 1 to 4 show that the call price is an increasing function of the excess performance α, and in turn drift μ, a result to be expected.

The call price is also increasing with the high water mark level H as incentive fees get triggered less often.

Table 5 was just meant to check the exactitude of our coding program : by choosing $a = 0$ and $\alpha = 0$, the NAV call option pricing problem is reduced to the Merton (1973) formula. Table 5 shows that the prices obtained by inversion of the Laplace transform are remarkably close to those provided by the Merton analytical formula.

Table 1

Call Option Prices at a volatility level $\sigma = 20\%$

$H = \$85$, $S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$14.5740	\$18.9619
100%	\$7.6175	\$12.1470
110%	\$3.3054	\$7.2058

$H = S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$ and $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$15.0209	\$19.6866
100%	\$7.8346	\$12.5922
110%	\$3.3837	\$7.4427

$H = \$115$, $S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$15.7095	\$20.8464
100%	\$8.4147	\$13.5815
110%	\$3.7084	\$8.1198

Table 2

Call Option Prices at a volatility level $\sigma = 20\%$

$H = \$85$, $S_0 = \$100$, $\alpha = 15\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 20\%$

Strike / Maturity	6 Months	1 Year
90%	\$16.3804	\$22.6562
100%	\$8.9668	\$15.1925
110%	\$4.1091	\$9.4795

$H = S_0 = \$100$, $\alpha = 15\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$ and $\mu = 20\%$

Strike / Maturity	6 Months	1 Year
90%	\$16.9611	\$23.6036
100%	\$9.2703	\$15.8190
110%	\$4.2276	\$9.8398

$H = \$115$, $S_0 = \$100$, $\alpha = 15\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 20\%$

Strike / Maturity	6 Months	1 Year
90%	\$17.9362	\$25.2503
100%	\$10.1156	\$17.2719
110%	\$4.7300	\$10.8943

Table 3

Call Option Prices at a volatility level $\sigma = 40\%$

$H = \$85$, $S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$18.8245	\$25.3576
100%	\$13.2042	\$19.9957
110%	\$8.9804	\$15.6276

$H = S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$ and $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$19.1239	\$25.8231
100%	\$13.3979	\$20.3534
110%	\$9.1012	\$15.8949

$H = \$115$, $S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$19.5128	\$26.4273
100%	\$13.7277	\$20.8726
110%	\$9.3409	\$16.3134

Table 4

Call Option Prices at a volatility level $\sigma = 40\%$

$H = \$85$, $S_0 = \$100$, $\alpha = 15\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 20\%$

Strike / Maturity	6 Months	1 Year
90%	\$20.3926	\$28.6499
100%	\$14.4928	\$22.8861
110%	\$9.9903	\$18.1179

$H = S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$ and $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$20.7978	\$29.2995
100%	\$14.7618	\$23.3938
110%	\$10.1615	\$18.5044

$H = \$115$, $S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$21.3417	\$30.1555
100%	\$15.2260	\$24.1402
110%	\$10.5042	\$19.1158

Table 5

NAV Call Option Prices when $\mu = 0$ at a volatility level $\sigma = 40\%$

$S_0 = \$100$, $r = 2\%$, $c = 0.3\%$

Maturity	6 months		1 year	
Strike	Laplace Transform	Merton formula	Laplace Transform	Merton formula
90%	\$12.3324	\$12.3324	\$14.577	\$14.577
100%	\$6.0375	\$6.0375	\$8.7434	\$8.7434
110%	\$2.4287	\$2.4287	\$4.8276	\$4.8276

IV Conclusion

In this paper, we proposed a pricing formula for options on hedge funds that accounts for the high-water mark rule defining the performance fees paid to the fund managers. The geometric Brownian motion dynamics chosen for the hedge fund Net Asset Value allowed us to exhibit an explicit expression of the Laplace transform in maturity of the option price through the use of Brownian local times. Numerical results obtained by inversion of these Laplace transforms display the influence of key parameters such as volatility or moneyness on the NAV call price.

References

[1] Abate, J. and W. Whitt, "Numerical Inversion of Laplace Transforms of Probability Distributions," *ORSA Journal on Computing*, 7 (1995), 36-43.

[2] Agarwal, V. and N.Y Naik, "Multi-period performance persistence analysis of hedge funds," *Journal of Financial and Quantitative Analysis*, 35 (2000), 3, 327-342.

[3] Amin G. and H. Kat, "Hedge Fund Performance 1990-2000: Do the "Money Machines" really Add Value?," *Journal of Financial and Quantitative Analysis*, 38 (2003), 2, 251-274.

[4] Black F. and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81 (1973), 637-654.

[5] Brown, S.J., W.N. Goetzmann and R.G. Ibbotson, "Offshore Hedge Funds: Survival and Performance, 1989-1995," *Journal of Business*, 72 (1999), 1, 91-117.

[6] Fung, W. and D.A. Hsieh, "A Primer on Hedge Funds," *Journal of Empirical Finance*, 6 (1999), 309-331.

[7] Geman, H. and C. Kharoubi, "Hedge Funds Revisited: Distributional Characteristics, Dependence Structure and Diversification," *Journal of Risk*, 5 (2003), 55-74.

[8] Geman, H. and M. Yor, "Pricing and Hedging Double Barrier Options : A Probabilistic Approach," *Mathematical Finance*, 6 (1996), 4, 365-378.

[9] Goetzmann, W.N., J. Ingersoll and S.A. Ross, "High-Water Marks and Hedge Fund Management Contracts," *Journal of Finance*, 58 (2003), 1685-1718.

[10] Jeanblanc, M., J. Pitman and M. Yor, "Feynman-Kac formula and decompositions of Brownian paths," *Computational and applied Mathematics*, 16 (1997), 27-52, Birkhäuser.

[11] Karatzas, I. and S.E. Shreve, *Brownian Motion and Stochastic Calculus, Second Edition*, Springer-Verlag, New York (1991).

[12] Kunitomo, N. and M. Ikeda, "Pricing Options with Curved Boundaries," *Mathematical Finance*, 2 (1992), 275-298.

[13] McKean, H.P., *Stochastic Integrals*, New York, Academic Press (1969).

[14] Merton, R.C., "Theory of Rational Option Pricing, " *Bell Journal of Economics and Management Science*, 4 (1973), 141-183.

[15] Revuz, D. and M. Yor, *Continuous Martingales and Brownian Motion, Third Edition*, Springer-Verlag, Berlin (2005).

[16] Walker, J.S., Fast Fourier Transform, 2nd ed. Boca Raton, CRC Press (1996).

[17] Zvonkin, A.K., "A Transformation of the phase space of a process that removes the drift," *Math. USSR Sbornik*, 2 (1974), 129-149.

Appendix : Excursion Theory

Proof of Proposition II.5: We use the Master formula exhibited in Brownian excursion theory; for more details see Chapter XII in Revuz and Yor (2005) the notation of which we borrow:
n denotes the Itô characteristic measure of excursions and n_+ is the restriction of n to positive excursions;
$V(\epsilon) = \inf\{t > 0; \epsilon(t) = 0\}$ for $\epsilon \in \mathbf{W}_{exc}$ the space of excursions,
$(\tau_l)_{l \geq 0}$ is the inverse local time of the Brownian motion.

We can write

$$\mathbb{E}\Big[\int_0^\infty dte^{-\frac{\theta}{2}t} h(W_t)\exp(\lambda L_t)\exp(-\mu A_t^+ - \nu A_t^-)\Big] = I \cdot J$$

where

$$\begin{aligned} I &= \mathbb{E}\big[\int_0^\infty dle^{-\frac{\theta}{2}\tau_l} e^{\lambda l}\exp(-\mu A_{\tau_l}^+ - \nu A_{\tau_l}^-)\big] \\ &= \int_0^\infty dl \exp\left(l\big(\lambda - \int n(d\epsilon)(1 - e^{-\frac{\theta}{2}V - \mu A_V^+ - \nu A_V^-})\big)\right) \\ &= \frac{1}{\int n(d\epsilon)\big(1 - e^{-\frac{\theta}{2}V - \mu A_V^+ - \nu A_V^-}\big) - \lambda} \\ &= \frac{1}{\frac{\sqrt{\theta+2\mu}+\sqrt{\theta+2\nu}}{2} - \lambda} \end{aligned}$$

and

$$J = \int_0^\infty dse^{-\frac{\theta}{2}s}\Big\{e^{-\mu s} n_+\big(h(\epsilon_s)\mathbf{1}_{\{s<V\}}\big) + e^{-\nu s} n_+\big(h(-\epsilon_s)\mathbf{1}_{\{s<V\}}\big)\Big\}$$

Next, we use the result

$$n_+\big(\epsilon_s \in dy; s < V\big) = \frac{y}{\sqrt{2\pi s^3}} e^{-\frac{y^2}{2s}} dy \qquad (y > 0) \tag{15}$$

and obtain

$$J = \int_0^\infty dxe^{-x\sqrt{\theta+2\mu}} h(x) + \int_0^\infty dxe^{-x\sqrt{\theta+2\nu}} h(-x) \tag{16}$$

where the proof of equation (16) comes from the fact that in (15) the density of n_+ as a function of s, is precisely the density of $T_y = \inf\{t : B_t = y\}$, and

$\mathbb{E}[e^{-\lambda T_y}] = e^{-y\sqrt{2\lambda}}$. ■

This example of application of excursion theory is one of the simplest illustrations of Feynman-Kac type computations which may be obtained with excursion theory arguments. For a more complete story, see Jeanblanc, Pitman and Yor (1997).